

Aluthge transforms of weighted shifts on directed trees

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ABSTRACT. Aluthge transform of a bounded operator is generalized to the case of unbounded one. A formula for the Aluthge transform of a weighted shift on a directed tree is established and it is used to construct an example of a hyponormal operator whose Aluthge transform has trivial domain. It is proven that such an example can be also constructed in the class of composition operators. It is also shown that Aluthge transform of a closed, densely defined operator is not necessarily closable.

1. Introduction

Aluthge transform of a bounded operator T , introduced by Aluthge in [1], is given by the formula $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of T . It turned out to have many applications, e.g. in the invariant subspace problem (cf. [9]). One of the most important properties of the Aluthge transform is that it transforms a p -hyponormal operator into a $(p + \frac{1}{2})$ -hyponormal one, preserving its spectrum (cf. [1], [5]). Moreover, under some conditions, the sequence $\{\tilde{T}^{(n)}\}$ of consecutive iterations of Aluthge transform is convergent to a normal operator (cf. [11]). Aluthge transforms of operators were studied also in [3], [4], [7], [10], [12].

A natural question is which of the above mentioned properties remain true if one considers a closed, densely defined operator T , which is not necessarily bounded. In this paper it is shown that Aluthge transform of such an operator may have trivial domain and need not be necessarily closed or even closable. Thus the sequence $\{\tilde{T}^{(n)}\}$ cannot be defined. What is interesting, \tilde{T} may have trivial domain even if T is a hyponormal operator, which implies in particular that Aluthge transform does not preserve hyponormality in the unbounded case. An example of such a hyponormal operator is given in this paper in the class of weighted shifts on directed trees. The construction of the example is preceded by a discussion on Aluthge transform for this class of operators.

What is important, the directed tree used in the construction is rootless and therefore the operator in question is unitarily equivalent to a composition operator. In turn, an example of an operator whose Aluthge transform is not closable can be realized as the adjoint of a composition operator.

2010 *Mathematics Subject Classification.* Primary 47B37; Secondary 47B20, 47B33.

Key words and phrases. Aluthge transform, weighted shift on a directed tree, hyponormal operator, composition operator on an L^2 -space, polar decomposition.

Since most of the properties of Aluthge transform is preserved if one replaces $\frac{1}{2}$ in its definition by any other exponents that sum up to 1 (cf. [2], [5]), in this paper t -Aluthge transform is considered for any $t \in (0, 1]$, according to the following definition:

DEFINITION 1.1. Let T be a closed, densely defined operator in a Hilbert space \mathcal{H} , let $T = U|T|$ be its polar decomposition and let $t \in (0, 1]$. Then t -Aluthge transform of T is given by the formula $\Delta_t(T) = |T|^t U |T|^{1-t}$.

2. Preliminaries

In what follows \mathbb{Z} will denote the set of all integers and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. For any set A the cardinality of A will be denoted by $\#A$.

Let T be any operator in a complex Hilbert space \mathfrak{H} . Then $\mathcal{D}(T)$, $\mathcal{N}(T)$, $\mathcal{R}(T)$ denote the domain, the null space and the range of T , respectively. For any linear subspace W of $\mathcal{D}(T)$ we denote by $T \upharpoonright_W$ the restriction of T to the subspace W . Let $\Gamma(T) \subset \mathcal{H} \times \mathcal{H}$ be the graph of T . If the closure of $\Gamma(T)$ in the product topology is a graph of an operator, we call this operator the closure of T and denote by \overline{T} .

A densely defined operator T is called *hyponormal*, if $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $\|Tf\| \geq \|T^*f\|$ for every $f \in \mathcal{D}(T)$.

Let $\mathfrak{T} = (V, E)$ be a directed tree (i.e. V and E are the sets of vertices and edges, respectively). If \mathfrak{T} has a root, we denote it by **root** and we set $V^\circ = V \setminus \{\mathbf{root}\}$. Otherwise, we set $V^\circ = V$. For any vertex $u \in V$ we put $\mathbf{Chi}(u) = \{v \in V : (u, v) \in E\}$. If $v \in V^\circ$, than by $\mathbf{par}(v)$ we denote the only vertex $u \in V$ such that $v \in \mathbf{Chi}(u)$.

By $\ell^2(V)$ we understand the complex Hilbert space of functions $f : V \rightarrow \mathbb{C}$ such that $\sum_{v \in V} |f(v)|^2 < \infty$, with inner product $\langle f, g \rangle = \sum_{v \in V} f(v) \overline{g(v)}$, $f, g \in \ell^2(V)$. For any $u \in V$ we define $e_u \in \ell^2(V)$ as follows:

$$e_u(v) = \begin{cases} 1, & u = v \\ 0, & u \neq v \end{cases}.$$

Obviously, $\{e_u\}_{u \in V}$ is an orthonormal basis of $\ell^2(V)$. We denote by \mathcal{E}_V the linear span of $\{e_u\}_{u \in V}$.

For any system $\lambda = \{\lambda_v\}_{v \in V^\circ}$ we define operator S_λ in $\ell^2(V)$ by

$$(2.1) \quad \mathcal{D}(S_\lambda) = \left\{ f \in \ell^2(V) : \sum_{u \in V} \left(\sum_{v \in \mathbf{Chi}(u)} |\lambda_v|^2 \right) |f(u)|^2 < \infty \right\},$$

$$(2.2) \quad (S_\lambda f)(v) = \begin{cases} \lambda_v f(\mathbf{par}(v)), & v \in V^\circ \\ 0, & v = \mathbf{root} \end{cases}, \quad f \in \mathcal{D}(S_\lambda).$$

The operator S_λ is called the *weighted shift* on the directed tree \mathfrak{T} with the system of weights λ .

For any $\lambda = \{\lambda_u\}_{u \in V^\circ}$ we will use the following notations: $V_\lambda^+ := \{u \in V : S_\lambda e_u \neq 0\}$, $\mathbf{Chi}_\lambda^+(u) := \mathbf{Chi}(u) \cap V_\lambda^+$ for any $u \in V$ and, if U is any subset of V , then $\mathbf{Chi}(U) := \bigcup_{u \in U} \mathbf{Chi}(u)$. We also use notatins $\mathbf{Chi}^2(u) := \mathbf{Chi}(\mathbf{Chi}(u))$ and $\mathbf{par}^2(u) = \mathbf{par}(\mathbf{par}(u))$.

Recall some useful properties of weighted shifts.

PROPOSITION 2.1 (cf. [8, Propositions 3.1.2 and 3.1.3]). *Let S_{λ} be a weighted shift on a directed tree $\mathfrak{T} = (V, E)$. Then the following assertions hold:*

- (i) S_{λ} is a closed operator,
- (ii) $e_u \in \mathcal{D}(S_{\lambda})$ if and only if $\sum_{v \in \text{Chi}(u)} |\lambda_v|^2 < \infty$ and in this case

$$(2.3) \quad S_{\lambda}e_u = \sum_{v \in \text{Chi}(u)} \lambda_v e_v, \quad \|S_{\lambda}e_u\|^2 = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2,$$

- (iii) S_{λ} is densely defined if and only if $e_u \in \mathcal{D}(S_{\lambda})$ for every $u \in V$.

LEMMA 2.2. *Let S_{λ} be a weighted shift on $\mathfrak{T} = (V, E)$. Then $\mathcal{E} := \mathcal{E}_V \cap \mathcal{D}(S_{\lambda})$ is a core for S_{λ} , i.e. $\overline{S_{\lambda} \upharpoonright_{\mathcal{E}}} = S_{\lambda}$.*

PROOF. Let $f \in \mathcal{D}(S_{\lambda})$ and let $U := \{u \in V : f(u) \neq 0\}$. Since $f \in \ell^2(V)$, the set U is at most countable. If U is finite, then $f \in \mathcal{E}_V$. Otherwise, set $U = \{u_1, u_2, \dots\}$ and

$$(2.4) \quad f_n := \sum_{j=1}^n f(u_j) e_{u_j}, \quad n = 1, 2, \dots$$

Obviously, $f_n \in \mathcal{E}_V$. Since $f \in \mathcal{D}(S_{\lambda})$, for every n we have

$$\sum_{u \in V} \left(\sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \right) |f_n(u)|^2 \leq \sum_{u \in V} \left(\sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \right) |f(u)|^2 < \infty$$

and therefore $f_n \in \mathcal{D}(S_{\lambda})$ and hence also $f_n \in \mathcal{E}$.

Using Parseval's identity, we get

$$\begin{aligned} \|f - f_n\|^2 &= \sum_{u \in V} |\langle f - f_n, e_u \rangle|^2 = \sum_{u \in V} |f(u) - f_n(u)|^2 = \\ &= \sum_{u \in U} |f(u) - f_n(u)|^2 = \sum_{j=n+1}^{\infty} |f(u_j)|^2 \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

because the series $\sum_{j=1}^{\infty} |f(u_j)|^2$ is convergent.

It remains only to show that $S_{\lambda}f_n \rightarrow S_{\lambda}f$. Using (2.4) and (2.3), we have

$$S_{\lambda}f_n = S_{\lambda} \sum_{j=1}^n f(u_j) e_{u_j} = \sum_{j=1}^n f(u_j) \left(\sum_{v \in \text{Chi}(u_j)} \lambda_v e_v \right).$$

Using Parseval's identity again, from (2.2) we get

$$\begin{aligned}
\|S_{\lambda}(f - f_n)\|^2 &= \sum_{v \in V} |\langle S_{\lambda}(f - f_n), e_v \rangle|^2 = \\
&= \sum_{v \in V} |(S_{\lambda}f)(v) - (S_{\lambda}f_n)(v)|^2 = \\
&= \sum_{v \in V^{\circ}} |\lambda_v|^2 |f(\text{par}(v)) - f_n(\text{par}(v))|^2 = \\
&= \sum_{u \in V} \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 |f(u) - f_n(u)|^2 = \\
&= \sum_{u \in U} |f(u) - f_n(u)|^2 \left(\sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \right) = \\
&= \sum_{j=n+1}^{\infty} |f(u_j)|^2 \left(\sum_{v \in \text{Chi}(u_j)} |\lambda_v|^2 \right) \xrightarrow{n \rightarrow \infty} 0,
\end{aligned}$$

which completes the proof. \square

Let us recall a criterion for hyponormality which will be used in the sequel.

THEOREM 2.3 (cf. [8, Theorem 5.1.2, Remark 5.1.5]). *Let S_{λ} be a weighted shift with weights λ on a directed tree $\mathfrak{T} = (V, E)$. Then S_{λ} is hyponormal if and only if for every $u \in V$ the following conditions hold:*

$$(2.5) \quad \text{if } v \in \text{Chi}(u) \text{ and } \|S_{\lambda}e_v\| = 0, \text{ then } \lambda_v = 0,$$

$$(2.6) \quad \sum_{v \in \text{Chi}_{\lambda}^{+}(u)} \frac{|\lambda_v|^2}{\|S_{\lambda}e_v\|^2} \leq 1.$$

3. Polar decompositions of S_{λ} and S_{λ}^*

We begin by recalling the description of the polar decomposition of a weighted shift.

PROPOSITION 3.1 (cf. [8, Proposition 3.4.3]). *Let S_{λ} be a densely defined weighted shift on $\mathfrak{T} = (V, E)$ with weights λ and let $\alpha > 0$. Then:*

- (i) $\mathcal{D}(|S_{\lambda}|^{\alpha}) = \{f \in \ell^2(V) : \sum_{u \in V} \|S_{\lambda}e_u\|^{2\alpha} |f(u)|^2 < \infty\}$,
- (ii) for every $u \in V$ we have $e_u \in \mathcal{D}(|S_{\lambda}|^{\alpha})$ and $|S_{\lambda}|^{\alpha}e_u = \|S_{\lambda}e_u\|^{\alpha}e_u$,
- (iii) if $f \in \mathcal{D}(|S_{\lambda}|^{\alpha})$, then

$$(3.1) \quad (|S_{\lambda}|^{\alpha}f)(u) = \|S_{\lambda}e_u\|^{\alpha}f(u), \quad u \in V.$$

PROPOSITION 3.2 (cf. [8, Proposition 3.5.1]). *Let S_{λ} be a densely defined weighted shift on $\mathfrak{T} = (V, E)$ with weights λ and let $S_{\lambda} = U|S_{\lambda}|$ be its polar decomposition. Then $U = S_{\pi}$, where*

$$(3.2) \quad \pi_v = \begin{cases} \frac{\lambda_v}{\|S_{\lambda}e_{\text{par}(v)}\|}, & \text{if } \text{par}(v) \in V_{\lambda}^{+}, \\ 0, & \text{otherwise} \end{cases}, \quad v \in V^{\circ}.$$

The following proposition contains a formula for S_λ^* .

PROPOSITION 3.3 (cf. [8, Proposition 3.4.1]). *Let S_λ be a densely defined weighted shift on a directed tree $\mathfrak{T} = (V, E)$. Then*

(i) $\mathcal{E}_V \subseteq \mathcal{D}(S_\lambda^*)$ and

$$S_\lambda^* e_u = \begin{cases} \overline{\lambda_u} e_{\text{par}(u)}, & u \in V^\circ \\ 0, & u = \text{root} \end{cases},$$

(ii) $\mathcal{D}(S_\lambda^*) = \left\{ f \in \ell^2(V) : \sum_{u \in V} \left| \sum_{v \in \text{Chi}(u)} \overline{\lambda_v} f(v) \right|^2 < \infty \right\},$

(iii) $(S_\lambda^* f)(u) = \sum_{v \in \text{Chi}(u)} \overline{\lambda_v} f(v)$ for every $u \in V$ and $f \in \mathcal{D}(S_\lambda^*)$.

Let $S_\lambda^* = V|S_\lambda^*|$ be the polar decomposition of S_λ^* . From Proposition 3.2 it follows that $V = S_\pi^*$ with π given by (3.2). The exact formula for S_π^* can be easily derived from Proposition 3.3.

The following theorem gives a formula for powers of modulus of S_λ^* .

THEOREM 3.4. *Let S_λ be a densely defined weighted shift on directed tree $\mathfrak{T} = (V, E)$ and let $\alpha > 0$. Then the following assertions hold:*

(i) $\mathcal{D}(|S_\lambda^*|^\alpha) = \left\{ f \in \ell^2(V) : \sum_{u \in V^\circ} \|S_\lambda e_u\|^{2\alpha-2} \left| \sum_{v \in \text{Chi}(u)} \overline{\lambda_v} f(v) \right|^2 < \infty \right\}.$

(ii) for every $u \in V$ the following formula holds:

$$(|S_\lambda^*|^\alpha f)(u) = \begin{cases} \|S_\lambda e_{\text{par}(u)}\|^{\alpha-2} \lambda_u \sum_{v \in \text{Chi}(\text{par}(u))} \overline{\lambda_v} f(v), & \text{if } u \in \text{Chi}(V_\lambda^+) \\ 0, & \text{if } u \in V \setminus \text{Chi}(V_\lambda^+) \end{cases},$$

(iii) the formula

$$(3.3) \quad |S_\lambda^*|^\alpha = \bigoplus_{u \in V} \|S_\lambda e_u\|^\alpha P_u$$

holds, where P_u is the orthogonal projection from $\ell^2(V)$ onto the linear span of $S_\lambda e_u$ for all $u \in V$ (if $S_\lambda e_u = 0$, then $P_u = 0$).

PROOF. We start by proving all assertions for $\alpha = 1$.

It is known that $\mathcal{D}(|S_\lambda^*|) = \mathcal{D}(S_\lambda^*)$, which together with (2.1), implies the part (i). Moreover, since $S_\lambda^* = S_\pi^* |S_\lambda^*|$ is the polar decomposition of S_λ^* , where π is given by (3.2), we conclude that $|S_\lambda^*| = S_\pi S_\lambda^*$. Hence for all $f \in \mathcal{D}(S_\lambda^*)$ and $u \in V$ we obtain

$$\begin{aligned} (|S_\lambda^*|f)(u) &= \begin{cases} \pi_u(S_\lambda^* f)(\text{par}(u)), & \text{if } u \in V^\circ \\ 0, & \text{if } u = \text{root} \end{cases} = \\ (3.3) \quad & \begin{cases} \frac{\lambda_u}{\|S_\lambda e_{\text{par}(u)}\|} \sum_{v \in \text{Chi}(\text{par}(u))} \overline{\lambda_v} f(v), & \text{if } u \in \text{Chi}(V_\lambda^+) \\ 0, & \text{if } u \in V \setminus \text{Chi}(V_\lambda^+) \end{cases}, \end{aligned}$$

so assertion (ii) holds for $\alpha = 1$.

Let now P_u be as in (iii). Then for every $f \in \ell^2(V)$, $u \in V_\lambda^+$ and $w \in V$ we have

$$\begin{aligned}
 P_u f &= \frac{\langle f, S_\lambda e_u \rangle S_\lambda e_u}{\|S_\lambda e_u\|^2} = \frac{1}{\|S_\lambda e_u\|^2} \left\langle f, \sum_{v \in \text{Chi}(u)} \lambda_v e_v \right\rangle \sum_{w \in \text{Chi}(u)} \lambda_w e_w = \\
 (3.4) \quad &= \frac{1}{\|S_\lambda e_u\|^2} \left(\sum_{v \in \text{Chi}(u)} \overline{\lambda}_v f(v) \right) \sum_{w \in \text{Chi}(u)} \lambda_w e_w.
 \end{aligned}$$

Observe that if $\langle P_u f, P_v g \rangle \neq 0$ for some $u, v \in V$ and $f, g \in \ell^2(V)$, then from (3.4) it follows that there exists $w \in \text{Chi}(u) \cap \text{Chi}(v)$. This implies that $u = \text{par}(w) = v$. Hence the orthogonality of the sum in (iii) follows.

For any $f \in \mathcal{D}(S_\lambda^*)$, $u \in V_\lambda^+$ and $w \in V$ we infer from (3.4) that

$$\begin{aligned}
 (P_u f)(w) &= \begin{cases} \frac{\lambda_w}{\|S_\lambda e_u\|^2} \sum_{v \in \text{Chi}(u)} \overline{\lambda}_v f(v), & \text{if } w \in \text{Chi}(u) \\ 0, & \text{if } w \in V \setminus \text{Chi}(u) \end{cases} = \\
 &= \begin{cases} \frac{\lambda_w}{\|S_\lambda e_{\text{par}(w)}\|^2} \sum_{v \in \text{Chi}(\text{par}(w))} \overline{\lambda}_v f(v), & \text{if } w \in \text{Chi}(u) \\ 0, & \text{if } w \in V \setminus \text{Chi}(u) \end{cases} = \\
 (3.5) \quad &= \begin{cases} \frac{1}{\|S_\lambda e_{\text{par}(w)}\|} (|S_\lambda^*|f)(w), & \text{if } w \in \text{Chi}(u) \\ 0, & \text{if } w \in V \setminus \text{Chi}(u) \end{cases}.
 \end{aligned}$$

Hence for all $f \in \mathcal{D}(S_\lambda^*)$ and $w \in V^\circ$ we obtain

$$(|S_\lambda^*|f)(w) = \|S_\lambda e_{\text{par}(w)}\| (P_{\text{par}(w)} f)(w) = \sum_{u \in V} \|S_\lambda e_u\| (P_u f)(w).$$

Therefore $|S_\lambda^*|f = \sum_{u \in V} \|S_\lambda e_u\| P_u f$ for every $f \in \mathcal{D}(S_\lambda^*)$. To show that

$$(3.6) \quad |S_\lambda^*| = \bigoplus_{u \in V} \|S_\lambda e_u\| P_u,$$

it now suffices to check the inclusion

$$\mathcal{D}\left(\bigoplus_{u \in V} \|S_\lambda e_u\| P_u\right) \subseteq \mathcal{D}(S_\lambda^*).$$

Let f belong to the left-hand side. This means that

$$\begin{aligned}
 \infty &> \sum_{u \in V} \|S_\lambda e_u\|^2 \|P_u f\|^2 \stackrel{(3.4)}{=} \\
 &= \sum_{u \in V} \|S_\lambda e_u\|^2 \left(\frac{1}{\|S_\lambda e_u\|^4} \left| \sum_{v \in \text{Chi}(u)} \overline{\lambda}_v f(v) \right|^2 \|S_\lambda e_u\|^2 \right) = \\
 (3.7) \quad &= \sum_{u \in V} \left| \sum_{v \in \text{Chi}(u)} \overline{\lambda}_v f(v) \right|^2,
 \end{aligned}$$

which implies that $f \in \mathcal{D}(S_\lambda^*) = \mathcal{D}(|S_\lambda^*|)$. This completes the prove of (3.6).

Let now $\alpha > 0$ be arbitrary. The assertion (iii) of the theorem follows immediately from (3.6). To prove (i) it now suffices to observe that, by calculations similar to (3.7), $f \in \mathcal{D}(|S_\lambda^*|^\alpha) = \mathcal{D}(\bigoplus_{u \in V} \|S_\lambda e_u\|^\alpha P_u)$ if and only if

$$\infty > \sum_{u \in V} \|S_\lambda e_u\|^{2\alpha} \|P_u f\|^2 = \sum_{u \in V} \|S_\lambda e_u\|^{2\alpha-2} \left| \sum_{v \in \text{Chi}(u)} \overline{\lambda}_v f(v) \right|^2.$$

Finally, let $f \in \mathcal{D}(|S_\lambda^*|^\alpha)$ and $w \in V$. Then

$$\begin{aligned} (|S_\lambda^*|^\alpha f)(w) &\stackrel{(iii)}{=} \sum_{u \in V} \|S_\lambda e_u\|^\alpha (P_u f)(w) = \\ &= \sum_{u \in V_\lambda^+} \|S_\lambda e_u\|^\alpha (P_u f)(w) = \\ &= \begin{cases} \|S_\lambda e_{\text{par}(w)}\|^\alpha (P_{\text{par}(w)} f)(w), & \text{if } w \in \text{Chi}(V_\lambda^+) \\ 0, & \text{if } w \in V \setminus \text{Chi}(V_\lambda^+) \end{cases} = \\ &\stackrel{(3.5)}{=} \begin{cases} \|S_\lambda e_{\text{par}(w)}\|^{2\alpha-2} \lambda_w \sum_{v \in \text{Chi}(\text{par}(w))} \overline{\lambda}_v f(v), & \text{if } w \in \text{Chi}(u) \\ 0, & \text{if } w \in V \setminus \text{Chi}(u) \end{cases}, \end{aligned}$$

which is exactly the claim of (ii). Thus the prove is complete. \square

4. Aluthge transform of a weighted shift

In this section we give a description of the t -Aluthge transform of a weighted shift on a directed tree. It turns out that its closure is again a weighed shift on the same tree.

THEOREM 4.1. *Let S_λ be a densely defined weighted shift on $\mathfrak{T} = (V, E)$ with $\lambda : V^\circ \rightarrow \mathbb{C}$ and let $t \in (0, 1]$. Then*

(i) $\mathcal{D}(\Delta_t(S_\lambda)) = \mathcal{D}(S_\mu) \cap \mathcal{D}(|S_\lambda|^{1-t})$, where

$$(4.1) \quad \mu_v = \begin{cases} \frac{\|S_\lambda e_v\|^t}{\|S_\lambda e_{\text{par}(v)}\|^t} \lambda_v, & \text{if } \text{par}(v) \in V_\lambda^+ \\ 0, & \text{otherwise} \end{cases}, \quad v \in V^\circ,$$

(ii) $\Delta_t(S_\lambda)$ is closable and $\overline{\Delta_t(S_\lambda)} = S_\mu$.

PROOF. Since $\Delta_t(S_\lambda) = |S_\lambda|^t S_\pi |S_\lambda|^{1-t}$, where π is given by (3.2), for any $f \in \ell^2(V)$ we have

$$(4.2) \quad f \in \mathcal{D}(\Delta_t(S_\lambda)) \iff (f \in \mathcal{D}(|S_\lambda|^{1-t}) \text{ and } S_\pi |S_\lambda|^{1-t} f \in \mathcal{D}(|S_\lambda|^t))$$

Let $f \in \mathcal{D}(|S_\lambda|^{1-t})$. Then, using (3.1) and (3.2), we obtain for every $v \in V^\circ$

$$\begin{aligned}
 (S_\pi |S_\lambda|^{1-t} f)(v) &= \pi_v(|S_\lambda|^{1-t} f)(\text{par}(v)) = \\
 &= \begin{cases} \frac{\lambda_v}{\|S_\lambda e_{\text{par}(v)}\|} \|S_\lambda e_{\text{par}(v)}\|^{1-t} f(\text{par}(v)), & \text{if } \text{par}(v) \in V_\lambda^+ \\ 0, & \text{otherwise} \end{cases} = \\
 (4.3) \quad &= \begin{cases} \frac{\lambda_v}{\|S_\lambda e_{\text{par}(v)}\|^t} f(\text{par}(v)), & \text{if } \text{par}(v) \in V_\lambda^+ \\ 0, & \text{otherwise} \end{cases}.
 \end{aligned}$$

From the above equation and Proposition 3.1 it follows that $f \in \mathcal{D}(\Delta_t(S_\lambda))$ if and only if

$$\begin{aligned}
 \infty &> \sum_{v \in V} \|S_\lambda e_v\|^{2t} |(S_\pi |S_\lambda|^{1-t} f)(v)|^2 = \\
 &= \sum_{v \in \text{Chi}(V_\lambda^+)} \|S_\lambda e_v\|^{2t} \left| \frac{\lambda_v}{\|S_\lambda e_{\text{par}(v)}\|^t} f(\text{par}(v)) \right|^2 = \\
 &= \sum_{u \in V_\lambda^+} |f(u)|^2 \sum_{v \in \text{Chi}(u)} \left| \frac{\|S_\lambda e_v\|^t}{\|S_\lambda e_{\text{par}(v)}\|^t} \lambda_v \right|^2 = \\
 &= \sum_{u \in V} |f(u)|^2 \sum_{v \in \text{Chi}(u)} |\mu_v|^2,
 \end{aligned}$$

which is equivalent to $f \in \mathcal{D}(S_\mu)$. This, due to (4.2), proves (i).

Let now $f \in \mathcal{D}(\Delta_t(S_\lambda))$. Then, using (3.1) and (4.3), we obtain

$$\begin{aligned}
 (\Delta_t(S_\lambda) f)(v) &= (|S_\lambda|^t S_\pi |S_\lambda|^{1-t} f)(v) = \\
 &= \|S_\lambda e_v\|^t (S_\pi |S_\lambda|^{1-t} f)(v) = \\
 &= \begin{cases} \frac{\|S_\lambda e_v\|^t}{\|S_\lambda e_{\text{par}(v)}\|^t} \lambda_v f(\text{par}(v)), & \text{if } \text{par}(v) \in V_\lambda^+ \\ 0, & \text{otherwise} \end{cases} = \\
 &= \mu_v f(\text{par}(v)) = (S_\mu f)(v),
 \end{aligned}$$

which proves that $\Delta_t(S_\lambda) \subseteq S_\mu$. Hence $\Delta_t(S_\lambda)$ is closable and $\overline{\Delta_t(S_\lambda)} \subseteq S_\mu$. But from Proposition 3.1 we know that $\mathcal{E}_V \subseteq \mathcal{D}(|S_\lambda|^{1-t})$. Part (ii) follows now from (i) and Lemma 2.2. \square

COROLLARY 4.2. *Let S_λ be a weighted shift on a directed tree $\mathfrak{T} = (V, E)$ and let $t \in (0, 1]$. Suppose there exists a constant $\alpha > 0$ such that $\|S_\lambda e_u\| \geq \alpha$ for every $u \in V$. Then $\Delta_t(S_\lambda) = S_\mu$, where μ is given by (4.1).*

PROOF. According to Theorem 4.1, it suffices to show that $\mathcal{D}(S_\mu) \subseteq \mathcal{D}(|S_\lambda|^{1-t})$. Let $f \in \mathcal{D}(S_\mu)$. Then

$$\begin{aligned} \infty &> \sum_{u \in V} \left(\sum_{v \in \text{Chi}(u)} |\mu_v|^2 \right) |f(u)|^2 = \\ &= \sum_{u \in V} \left(\sum_{v \in \text{Chi}(u)} \frac{\|S_\lambda e_v\|^{2t}}{\|S_\lambda e_{\text{par}(v)}\|^{2t}} |\lambda_v|^2 \right) |f(u)|^2 \geq \\ &\geq \sum_{u \in V} \left(\sum_{v \in \text{Chi}(u)} \frac{\alpha^{2t}}{\|S_\lambda e_u\|^{2t}} |\lambda_v|^2 \right) |f(u)|^2 = \\ &= \alpha^{2t} \sum_{u \in V} \frac{\|S_\lambda e_u\|^2}{\|S_\lambda e_u\|^{2t}} |f(u)|^2 = \alpha^{2t} \sum_{u \in V} \|S_\lambda e_u\|^{2-2t} |f(u)|^2, \end{aligned}$$

thus $f \in \mathcal{D}(|S_\lambda|^{1-t})$. \square

REMARK 4.3. If $t = 1$, then $\mathcal{D}(|S_\lambda|^{1-t}) = \mathcal{D}(I) = \ell^2(V)$. Hence $\Delta_1(S_\lambda) = S_\mu$, where μ is given by (4.1). This is in general not true for $t \in (0, 1)$, which is shown by the following example.

EXAMPLE 4.4. Let $t \in (0, 1)$ and $\mathfrak{T} = (V, E)$, where $V = \mathbb{N} = \{0, 1, \dots\}$ and $E = \{(n, n+1) : n \in \mathbb{N}\}$. For any $f \in \ell^2(\mathbb{N})$ such that $f(2k) \neq 0$ for each $k \in \mathbb{N}$, let

$$\begin{aligned} \lambda_{2k} &= 0, \\ \lambda_{2k+1} &= |f(2k)|^{\frac{1}{t-1}} \end{aligned}$$

for all $k \in \mathbb{N}$. Then $\mu_n = 0$ for every $n \in \mathbb{N}$ and therefore $\mathcal{D}(S_\mu) = \ell^2(V)$. But $f \notin \mathcal{D}(|S_\lambda|^{1-t})$, because

$$\begin{aligned} \sum_{u \in V} \|S_\lambda e_u\|^{2-2t} |f(u)|^2 &= \sum_{n=0}^{\infty} |\lambda_{n+1}|^{2-2t} |f(n)|^2 = \\ &= \sum_{k=0}^{\infty} |\lambda_{2k+1}|^{2-2t} |f(2k)|^2 = \sum_{k=0}^{\infty} |f(2k)|^{\frac{2-2t}{t-1}} |f(2k)|^2 = \sum_{k=0}^{\infty} 1 = \infty. \end{aligned}$$

Hence $\Delta_t(S_\lambda) \subsetneq S_\mu$. This example shows in particular that $\Delta_t(S_\lambda)$ may not be closed.

5. Aluthge transform of S_λ^*

The following theorem provides a formula for the t -Aluthge transform of the adjoint of a weighted shift.

THEOREM 5.1. Let S_λ be a densely defined weighted shift on a directed tree $\mathfrak{T} = (V, E)$ and let $t \in (0, 1]$. Then $\mathcal{E}_V \subseteq \mathcal{D}(\Delta_t(S_\lambda^*))$ and

$$\Delta_t(S_\lambda^*)e_v = \begin{cases} \frac{1}{\lambda_v} \frac{|\pi_{\text{par}(v)}|^2}{\mu_{\text{par}(v)}} S_\lambda e_{\text{par}^2(v)} & \text{if } v \in \text{Chi}^2(V_\lambda^+) \\ 0 & \text{if } v \in V \setminus \text{Chi}^2(V_\lambda^+) \end{cases},$$

where $\mu = \{\mu_w\}_{w \in V^\circ}$ and $\pi = \{\pi_w\}_{w \in V^\circ}$ are given by (4.1) and (3.2) respectively.

PROOF. Let $u, v \in V$ be any vertices and let P_u be as in Theorem 3.4. Then, by (3.4),

$$P_u e_v = \begin{cases} \frac{\overline{\lambda_v}}{\|S_{\lambda} e_u\|^2} \sum_{w \in \text{Chi}(u)} \lambda_w e_w, & \text{if } v \in \text{Chi}(u) \\ 0 & \text{if } v \in V \setminus \text{Chi}(u). \end{cases}$$

Hence, from Theorem 3.4 (iii) it follows for every $\alpha > 0$ that $e_v \in \mathcal{D}(|S_{\lambda}^*|^{\alpha})$ and the following equality holds:

$$(5.1) \quad |S_{\lambda}^*|^{\alpha} e_v = \begin{cases} \frac{\overline{\lambda_v}}{\|S_{\lambda} e_{\text{par}(v)}\|^{2-\alpha}} \sum_{w \in \text{Chi}(\text{par}(v))} \lambda_w e_w, & \text{if } v \in V^{\circ}, \\ 0, & \text{if } v = \text{root}. \end{cases}$$

Let now $S_{\lambda}^* = U|S_{\lambda}^*|$ be the polar decomposition of S_{λ}^* . Then, by Proposition 3.2, $U = S_{\pi}^*$, where $\pi = \{\pi_u\}_{u \in V}$ is given by (3.2). From Proposition 3.3 it follows that for every $w \in V^{\circ}$

$$(5.2) \quad S_{\pi}^* e_w = \overline{\pi_w} e_{\text{par}(w)} = \frac{\overline{\lambda_w}}{\|S_{\lambda} e_{\text{par}(w)}\|} e_{\text{par}(w)}.$$

Take $u \in V$. From (5.2) we obtain

$$\begin{aligned} \sum_{w \in \text{Chi}(u)} \|\lambda_w S_{\pi}^* e_w\|^2 &= \sum_{w \in \text{Chi}(u)} \frac{|\lambda_w|^4}{\|S_{\lambda} e_u\|^2} \leq \left(\sum_{w \in \text{Chi}(u)} \frac{|\lambda_w|^2}{\|S_{\lambda} e_u\|} \right)^2 = \\ &= \left(\frac{\|S_{\lambda} e_u\|^2}{\|S_{\lambda} e_u\|} \right)^2 = \|S_{\lambda} e_u\|^2 \end{aligned}$$

Hence the series $\sum_{w \in \text{Chi}(u)} \lambda_w S_{\pi}^* e_w$ is convergent in $\ell^2(V)$ and by (5.2)

$$\begin{aligned} \sum_{w \in \text{Chi}(u)} \lambda_w S_{\pi}^* e_w &= \sum_{w \in \text{Chi}(u)} \frac{|\lambda_w|^2}{\|S_{\lambda} e_{\text{par}(w)}\|} e_{\text{par}(w)} = \\ (5.3) \quad &= \sum_{w \in \text{Chi}(u)} \frac{|\lambda_w|^2}{\|S_{\lambda} e_u\|} e_u = \|S_{\lambda} e_u\| e_u. \end{aligned}$$

Since the series $\sum_{w \in \text{Chi}(u)} \lambda_w e_w = S_{\lambda} e_u$ is also convergent and S_{π}^* is a closed operator, it follows from (5.1) and (5.3) that $|S_{\lambda}^*|^t e_v \in \mathcal{D}(S_{\pi}^*)$ for any $v \in V^{\circ}$ and

$$\begin{aligned} S_{\pi}^* |S_{\lambda}^*|^{1-t} e_v &= \frac{\overline{\lambda_v}}{\|S_{\lambda} e_{\text{par}(v)}\|^{2-(1-t)}} S_{\pi}^* \left(\sum_{w \in \text{Chi}(\text{par}(v))} \lambda_w e_w \right) = \\ &= \frac{\overline{\lambda_v}}{\|S_{\lambda} e_{\text{par}(v)}\|^{1+t}} \sum_{w \in \text{Chi}(\text{par}(v))} \lambda_w S_{\pi}^* e_w = \\ &= \frac{\overline{\lambda_v}}{\|S_{\lambda} e_{\text{par}(v)}\|^t} e_{\text{par}(v)} \end{aligned}$$

Using (5.1) again, we get for any $v \in \text{Chi}(V^\circ)$

$$\begin{aligned} |S_\lambda^*|^t |S_\pi^*| |S_\lambda^*|^{1-t} e_v &= \\ &= \frac{\overline{\lambda_v}}{\|S_\lambda e_{\text{par}(v)}\|^t} \cdot \frac{\overline{\lambda_{\text{par}(v)}}}{\|S_\lambda e_{\text{par}^2(v)}\|^{2-t}} \sum_{w \in \text{Chi}(\text{par}^2(v))} \lambda_w e_w = \\ &= \frac{|\pi_{\text{par}(v)}|^2}{\overline{\lambda_v} \mu_{\text{par}(v)}} S_\lambda e_{\text{par}^2(v)} \end{aligned}$$

and clearly $|S_\lambda^*|^{1-t} |S_\pi^*| |S_\lambda^*|^t e_v = 0$ for every $v \in \text{Chi}(\text{root}) \cup \{\text{root}\}$. \square

6. An example of an operator with trivial Aluthge transform

In this section we construct a weighted shift S_λ with the following properties: S_λ is densely defined, injective and hyponormal, while $\mathcal{D}(\Delta_t(S_\lambda)) = \{0\}$ for every $t \in (0, 1]$ and $\Delta_t(S_\lambda^*)$ is not closable for any $t \in (0, 1)$. We also show that such an example can be constructed in the class of composition operators.

For any sequence $v : \mathbb{Z} \rightarrow \mathbb{Z}_+ \cup \{-1\}$ we define

$$\begin{aligned} m(v) &:= \inf\{n \in \mathbb{Z} : v_n \neq 0\} - 2, \\ M(v) &:= \sup\{n \in \mathbb{Z} : v_n > -1\}. \end{aligned}$$

Let

$$(6.1) \quad V = \{v : \mathbb{Z} \rightarrow \mathbb{Z}_+ \cup \{-1\} : m(v) > -\infty, M(v) < \infty \text{ and } v_n > -1 \text{ for } n \leq M(v)\},$$

$$(6.2) \quad E = \{(u, v) \in V \times V : M(v) = M(u) + 1 \text{ and } u_n = v_n \text{ for } n \leq M(u)\}.$$

Then $\mathfrak{T} = (V, E)$ is a rootless directed tree, such that for every $u \in V$ the set $\text{Chi}(u)$ is countable. Vertices of V are sequences of the form

$$u = (\dots, 0, 0, u_{m(u)}, \dots, u_{M(u)-1}, u_{M(u)}, -1, -1, \dots)$$

and for a vertex u given by the above formula we have

$$\begin{aligned} \text{par}(u) &= (\dots, 0, u_{m(u)}, \dots, u_{M(u)-1}, -1, -1, -1, \dots), \\ \text{Chi}(u) &= \{(\dots, 0, u_{m(u)}, \dots, u_{M(u)-1}, u_{M(u)}, n, -1, \dots) : n \in \mathbb{Z}_+\}. \end{aligned}$$

For any $v = (\dots, 0, v_{m(v)}, \dots, v_{M(v)}, -1, \dots) \in V$ let

$$(6.3) \quad \lambda_v = \frac{2^{v_{m(v)} + \dots + v_{M(v)} - 1}}{v_{M(v)} + 1}$$

In this section S_λ will always stand for the weighted shift on \mathfrak{T} with weights given by (6.3).

We start by proving that S_λ is densely defined. This follows from Proposition 2.1 and the following:

PROPOSITION 6.1. *For every $u \in V$, $e_u \in \mathcal{D}(S_\lambda)$ and*

$$\|S_\lambda e_u\| = 2^{u_{m(u)} + \dots + u_{M(u)}} \gamma,$$

where $\gamma = (\sum_{n=1}^{\infty} n^{-2})^{\frac{1}{2}}$.

PROOF. From (6.3) we get

$$\begin{aligned} \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 &= \sum_{v \in \text{Chi}(u)} \frac{2^{2(v_{m(v)} + \dots + v_{M(v)-1})}}{(v_{M(v)} + 1)^2} = \\ &= \sum_{v \in \text{Chi}(u)} \frac{2^{2(u_{m(u)} + \dots + u_{M(u)})}}{(v_{M(v)} + 1)^2} = 2^{2(u_{m(u)} + \dots + u_{M(u)})} \gamma^2. \end{aligned}$$

The claim follows now from Proposition 2.1 (ii). \square

To show hyponormality of S_λ we use Theorem 2.3.

PROPOSITION 6.2. *Operator S_λ is hyponormal.*

PROOF. From Proposition 6.1 it follows that $\|S_\lambda e_v\| > 0$ for every $v \in V$, so (2.5) is satisfied trivially. As for (2.6), for any $u \in V$ we have $\text{Chi}_\lambda^+(u) = \text{Chi}(u)$ and

$$\begin{aligned} \sum_{v \in \text{Chi}(u)} \frac{|\lambda_v|^2}{\|S_\lambda e_v\|^2} &= \sum_{v \in \text{Chi}(u)} \frac{2^{2(v_{m(v)} + \dots + v_{M(v)-1})}}{(v_{M(v)} + 1)^2 \cdot 2^{2(v_{m(v)} + \dots + v_{M(v)})} \gamma^2} = \\ &= \frac{1}{\gamma^2} \sum_{v \in \text{Chi}(u)} \frac{1}{(v_{M(v)} + 1)^2 2^{2v_{M(v)}}} < 1, \end{aligned}$$

because

$$\sum_{v \in \text{Chi}(u)} \frac{1}{(v_{M(v)} + 1)^2 2^{2v_{M(v)}}} < \sum_{v \in \text{Chi}(u)} \frac{1}{(v_{M(v)} + 1)^2} = \gamma^2.$$

This completes the proof. \square

In turn we show that the Aluthge transform of S_λ has trivial domain. Moreover, t -Aluthge transform of S_λ has trivial domain for arbitrarily small t .

PROPOSITION 6.3. *For any $t \in (0, 1]$ the domain of $\Delta_t(S_\lambda)$ is $\{0\}$.*

PROOF. Let $t \in (0, 1]$. From Theorem 4.1 and Proposition 6.1 we get $\Delta_t(S_\lambda) \subseteq S_\mu$, where

$$\begin{aligned} \mu_v &= \frac{\|S_\lambda e_v\|^t}{\|S_\lambda e_{\text{par}(v)}\|^t} \lambda_v = \\ &= \frac{2^{t(v_{m(v)} + \dots + v_{M(v)})} \gamma^t}{2^{t(v_{m(v)} + \dots + v_{M(v)-1})} \gamma^t} \cdot \frac{2^{v_{m(v)} + \dots + v_{M(v)-1}}}{v_{M(v)} + 1} = \\ (6.4) \quad &= \frac{2^{v_{m(v)} + \dots + v_{M(v)-1} + tv_{M(v)}}}{v_{M(v)} + 1}. \end{aligned}$$

Hence for any $u \in V$ we have

$$\begin{aligned} \sum_{v \in \text{Chi}(u)} |\mu_v|^2 &= \sum_{v \in \text{Chi}(u)} \frac{2^{2(v_{m(v)} + \dots + v_{M(v)-1} + tv_{M(v)})}}{(v_{M(v)} + 1)^2} = \\ &= 2^{2(u_{m(u)} + \dots + u_{M(u)})} \sum_{v \in \text{Chi}(u)} \frac{2^{2tv_{M(v)}}}{(v_{M(v)} + 1)^2} = \infty, \end{aligned}$$

and therefore, by Proposition 2.1, $e_u \notin \mathcal{D}(S_\mu)$. The claim follows now from Lemma 2.2. \square

The fact that $\Delta_t(S_\lambda^*)$ is not closable will follow from the lemma below:

LEMMA 6.4. *For any $t \in (0, 1)$ the operator $\Delta_t(S_\lambda^*)$ is densely defined and*

$$\mathcal{D}(\Delta_t(S_\lambda^*)^*) = \mathcal{N}(\Delta_t(S_\lambda^*)^*) = \mathcal{N}(S_\lambda^*).$$

PROOF. By Theorem 5.1, $\mathcal{E}_V \subseteq \mathcal{D}(\Delta_t(S_\lambda^*))$ and obviously \mathcal{E}_V is dense in $\ell^2(V)$. Moreover, since $\text{Chi}^2(V_\lambda^+) = V$, we have for every $v \in V$

$$(6.5) \quad \Delta_t(S_\lambda^*)e_v = \overline{\lambda_v} \frac{|\pi_{\text{par}(v)}|^2}{\mu_{\text{par}(v)}} S_\lambda e_{\text{par}^2(v)}.$$

Let $v = (\dots, 0, v_{m(v)}, \dots, v_{M(v)}, -1, \dots)$. From (6.3) and (6.4) we obtain

$$(6.6) \quad \begin{aligned} \frac{\overline{\lambda_v}}{\mu_{\text{par}(v)}} &= \frac{2^{v_{m(v)} + \dots + v_{M(v)} - 1}}{v_{M(v)} + 1} \frac{v_{M(v)-1} + 1}{2^{v_{m(v)} + \dots + v_{M(v)} - 2 + tv_{M(v)-1}}} = \\ &= \frac{v_{M(v)-1} + 1}{v_{M(v)} + 1} 2^{(1-t)v_{M(v)-1}}. \end{aligned}$$

In turn, by (3.2) and (2.3) we have

$$(6.7) \quad \begin{aligned} |\pi_{\text{par}(v)}|^2 &= \frac{|\lambda_{\text{par}(v)}|^2}{\|S_\lambda e_{\text{par}^2(v)}\|^2} = \\ &= \frac{2^{2(v_{m(v)} + \dots + v_{M(v)} - 2)}}{(v_{M(v)-1} + 1)^2 2^{2(v_{m(v)} + \dots + v_{M(v)} - 2)} \gamma^2} = \\ &= \frac{1}{(v_{M(v)-1} + 1)^2 \gamma^2}. \end{aligned}$$

Combining (6.5), (6.6) and (6.7) leads to the equality

$$\Delta_t(S_\lambda^*)e_v = \frac{2^{(1-t)v_{M(v)-1}}}{(v_{M(v)-1} + 1)(v_{M(v)} + 1)\gamma^2} \sum_{w \in \text{Chi}(\text{par}^2(v))} \lambda_w e_w.$$

Let $f \in \mathcal{D}(\Delta_t(S_\lambda^*)^*)$. Then for any $v \in V$

$$(6.8) \quad \begin{aligned} (\Delta_t(S_\lambda^*)^* f)(v) &= \langle \Delta_t(S_\lambda^*)^* f, e_v \rangle = \langle f, \Delta_t(S_\lambda^*)e_v \rangle = \\ &= \frac{2^{(1-t)v_{M(v)-1}}}{(v_{M(v)-1} + 1)(v_{M(v)} + 1)\gamma^2} \sum_{w \in \text{Chi}(\text{par}^2(v))} \overline{\lambda_w} f(w) = \\ &= \frac{2^{(1-t)v_{M(v)-1}}}{(v_{M(v)-1} + 1)(v_{M(v)} + 1)\gamma^2} (S_\lambda^* f)(\text{par}^2(v)). \end{aligned}$$

This gives the inclusion $\mathcal{N}(S_\lambda^*) \subseteq \mathcal{N}(\Delta_t(S_\lambda^*)^*)$. It suffices to show that $\mathcal{D}(\Delta_t(S_\lambda^*)^*) \subseteq \mathcal{N}(S_\lambda^*)$.

Suppose there exists $f \in \mathcal{D}(\Delta_t(S_\lambda^*)^*)$ such that $S_\lambda^* f \neq 0$. Let

$$u = (\dots, 0, u_{m(u)}, \dots, u_{M(u)}, -1, \dots) \in V$$

be such that $(S_{\lambda}^* f)(u) \neq 0$. Let $v^{(k)} = (\dots, 0, u_{m(u)}, \dots, u_{M(u)}, k, 0, -1, \dots)$ for every $k \in \mathbb{Z}_+$. Then $\text{par}^2(v^{(k)}) = u$ and

$$\begin{aligned} \|\Delta_t(S_{\lambda}^*)^* f\|^2 &\geq \sum_{k=0}^{\infty} |(\Delta_t(S_{\lambda}^*)^* f)(v^{(k)})|^2 = \\ &\stackrel{(6.8)}{=} \sum_{k=0}^{\infty} \frac{2^{2(1-t)k}}{(k+1)^{2\gamma^2}} |(S_{\lambda}^* f)(u)|^2 = \infty, \end{aligned}$$

because $t < 1$. This is a contradiction. Thus $S_{\lambda}^* f = 0$, which completes the proof. \square

COROLLARY 6.5. *Operator $\Delta_t(S_{\lambda}^*)$ is not closable for any $t \in (0, 1)$.*

PROOF. Since S_{λ}^* is a non-zero closed operator, $\mathcal{D}(\Delta_t(S_{\lambda}^*)^*) = \mathcal{N}(S_{\lambda}^*)$ is not dense in $\ell^2(V)$, which completes the proof. \square

By [7, Lemma 4.3.1], every weighted shift on a rootless directed tree with nonzero weights is unitarily equivalent to a composition operator in an L^2 -space over a σ -finite measure. From this, together with Propositions 6.1, 6.2, 6.3 and Corollary 6.5, we obtain the following theorem.

THEOREM 6.6. *There exists a hyponormal composition operator C in an L^2 -space over a σ -finite measure such that $\mathcal{D}(\Delta_t(C)) = \{0\}$ for $t \in (0, 1]$ and $\Delta_t(C^*)$ is not closable for $t \in (0, 1)$.*

REMARK 6.7. For any $u \in V$ let $W := \text{Des}(u) = \bigcup_{n=0}^{\infty} \text{Chi}^n(u)$ and let $\lambda' = \{\lambda_v\}_{v \in W \setminus \{u\}}$. Then $S_{\lambda'}$ is a weighted shift on a directed tree with root u . Moreover, $S_{\lambda'}$ has all properties claimed for S_{λ} , i.e. $S_{\lambda'}$ is densely defined, injective and hyponormal, its t -Aluthge transform has trivial domain for $t \in (0, 1]$ and t -Aluthge transform of $S_{\lambda'}^*$ is not closable for $t \in (0, 1)$. These assertions can be shown by repeating the proofs of all results from this section with appropriate changes.

It turns out that the tree given by (6.1) and (6.2) and the one described by Remark 6.7 are the only directed trees on which such an example can be constructed. This fact is stated in the following proposition.

PROPOSITION 6.8. *Let $\mathfrak{T} = (V, E)$ and $\lambda = \{\lambda_u\}_{u \in V^\circ} \subseteq \mathbb{C} \setminus \{0\}$. Suppose the weighted shift S_{λ} is densely defined and $\mathcal{D}(\Delta_t(S_{\lambda})) = \{0\}$ for some $t \in (0, 1]$. Then $\#\text{Chi}(u) = \aleph_0$ for every $u \in V$.*

PROOF. Let $u \in V$. Due to Proposition 2.1, $\overline{\mathcal{D}(S_{\lambda})} = \ell^2(V)$ implies that for $u \in V$ we have $e_u \in \mathcal{D}(S_{\lambda})$ and

$$\|S_{\lambda} e_u\|^2 = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2.$$

Since $|\lambda_v|^2 > 0$ for every $v \in \text{Chi}(u)$ and the above series is convergent, it follows that $\#\text{Chi}(u) \leq \aleph_0$.

Let $t \in (0, 1]$ be such that $e_u \notin \mathcal{D}(\Delta_t(S_{\lambda}))$. By Theorem 4.1, $\mathcal{D}(\Delta_t(S_{\lambda})) = \mathcal{D}(S_{\mu}) \cap \mathcal{D}(|S_{\lambda}|^{1-t})$, where μ is given by (4.1). Since $e_u \in \mathcal{D}(S_{\lambda}) \subseteq \mathcal{D}(|S_{\lambda}|^{1-t})$, it follows that $e_u \notin \mathcal{D}(S_{\mu})$. Hence

$$\infty = \sum_{w \in V} \left(\sum_{v \in \text{Chi}(w)} |\mu_v|^2 \right) |e_u(w)|^2 = \sum_{v \in \text{Chi}(u)} |\mu_v|^2,$$

which is possible only if $\#\text{Chi}(u) \geq \aleph_0$. This completes the proof. \square

A similar result with S_λ^2 instead of $\Delta_t(S_\lambda)$ was obtained in [6].

Acknowledgements. I would like to thank my supervisor, prof. Jan Stochel for encouragement and motivation, as well as substantial help he provided me while working on this paper.

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